A TORSIONAL FOURIER TRANSFORMER

John F. Wester

Letter on front cover:

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A TORSIONAL FOURIER TRANSFORMER

by

John F. Wester, Lieutenant, USN

B.S., U.S. Naval Academy, 1944

SUBMITTED IN PARTIAL FULFILLMENT OF THE

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1954

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A TORSIONAL FOURIER TRANSFORMER

by

John F. Wester

Submitted to the Department of Aeronautical Engineering on May 24, 1954, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis presents the results of an investigation into a method for obtaining a physical representation of Fourier transform of an arbitrary function by means of a torsional device.

A comparison of such a device is made with two other machines known to be in use for obtaining such a representation. It is shown that the capabilities of a torsional type transformer are sufficient to justify further investigation including the design and construction of a practical machine. The basic advantages of a torsional type transformer are simplicity, low cost, and inherent accuracy.

An investigation of errors arising from the approximation of Fourier transforms of certain functions by a step-wise summation instead of a continuous integral process is included. The results are presented for special cases of interest.

Thesis Supervisor: Sidney Lees

Title: Assistant Professor of Aeronautical Engineering

May 24, 1954

Professor Leicester F. Hamilton
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

Dear Professor Hamilton:

In accordance with the regulations of the faculty, I hereby submit a thesis entitled A Torsional Fourier Transformer in partial fulfillment of the requirements for the degree of Master of Science in Aeronautical Engineering.

ACKNOWLEDGEMENT

The author expresses his appreciation to the personnel of the Instrumentation Laboratory, Massachusetts Institute of Technology, who assisted in the preparation of this thesis. Particular gratitude is due Prof. Sidney Lees, who first conceived the principle of a torsional transformer, and who, as thesis supervisor, gave aid and encouragement during the weeks of experimentation.

Dr. J.H. Laning, Jr., of the Computing Section of the Laboratory, is responsible for much of the section on error analysis.

Mr. Paul Schaefer gave much needed advice on the mechanical design of the experimental equipment as well as constructing several of the more critical parts.

Mr. K. Tsutsumi, of Jackson & Moreland provided advice and encouragement on the design details of a practical device.

Mr. William Beale and Mr. Caesar Infascelli performed the necessary developing and enlarging, as well as furnishing advice on the photographic details of the investigation.

The graduate work for which this thesis is a partial requirement was performed while the author was assigned to the Massachusetts Institute of Technology by the U. S. Naval Bureau of Aeronautics.

TABLE OF CONTENTS

OBJECT

The object of this thesis is to investigate a method for obtaining the Fourier transform of an arbitrary function by means of a torsional device.

CHAPTER 1

INTRODUCTION

The development of complex physical systems has many phases and sub-phases before the final product is achieved. A brief description of these phases, and the inter-relationships between them, may be found in the introduction to Volume I of <u>Instrument Engineering</u> (1). Much of the information necessary for a successful end product must be generated during the actual development. This information may be broad in scope and variety, especially in new product developments. It may range from physical dimensions and dimensional tolerances to performance characteristics of components or the overall system. This thesis describes a device which can be useful in certain processes often used to obtain performance characteristics experimentally.

Testing of equipment to obtain actual performance characteristics is necessary for several reasons. Although performance characteristics may be predicted from theory, experimental verification is needed. In other situations testing may prove to be the only means of obtaining the necessary design information.

To this end, transient tests have been performed for many years(2, 3). During the early history of closed chain systems correct interpretation of the transient responses was difficult. To aid in this interpretation frequency analysis methods were borrowed from the practices of

communication engineering (4, 5, 6, 7). These frequency analysis methods include, as the experimental technique, the frequency response method. This procedure consists of recording the response of the system under test to a sinusoidally varying input. Comparison is then made of the steady-state amplitude and phase relationships existing between the input and the output. The tests are repeated for the range of frequencies desired to obtain the response characteristics as a function of frequency.

It is often inconvenient, and sometimes impossible, to perform steady-state sinusoidal frequency tests. An important parameter in aircraft dynamic stability calculations is the density altitude at which the tests are to be conducted. Hence the frequency method as applied to an aircraft in a gliding maneuver would introduce at least one extra variable into the test data. Certain components of expendable missiles, whose designed working life is only a few minutes, could not be tested satisfactorily by tests which would take hours to cover the desired frequency range. Other limitations of the sinusoidal frequence tests include the possibility of saturation of the equipment in the upper frequency range.

It has been shown by Bromberg (8), Seamans, Blasingame and Clementson (9) that the frequency response characteristics can also be obtained by considering the transient response of the system, particularly for a pulse function input. Floyd (10) and Bromberg (8) have described the inverse process of obtaining the transient response for any forcing function from the known steady-state sinusoidal frequency response.

The relating function, as defined by Draper, McKay and Lees (1) concerns the interrelationship between the steady-state frequency response, impulse function transient response, and the linear differential

equation which is associated with the physical system. The relating function concept states that these mathematical notions represent different ways to describe the basic functional dependence between two variables. Special cases of the relating function have been previously identified under various names by other authors (4, 6, 11, 12).

The relating function may be given in several forms. One such form is in terms of the relationship existing in the time domain between the input variation and the output variation. The relating function may also be expressed in terms of the relationship existing between these two variables in the frequency domain. When both the input and output variations possess zero initial conditions the relating function may be defined as the ratio of a response function to a forcing function. The response and forcing functions may be obtained as the Fourier transforms of the time-dependent response and input functions, respectively.

Fourier transforms have been derived for many common functions and are used extensively in design work. When the input and output variations are not available as analytic expressions one of two methods must be employed. Either the function must be approximated by a function or series of functions for which the transforms are known, or an exact method must be found for obtaining the relationship between the time and frequency dependent functions.

The first method presents several possibilities. The function may be approximated by a series of rectangular pulse functions of varying amplitudes. These pulses are of equal duration and occur at successive increments of time. The pulse duration must equal the time increment between initiation of successive pulses. Since the transform of the rectangular pulse function has been tabulated, the approximate transform of the original function may be found by a summation process.

Series of trapezodial functions have also been used to approximate the original function. These methods and their use are discussed in detail in Instrument Engineering, Chapter 25 (1).

Many sets of orthogonal functions may be used to describe variations found in nature (13). Of these the most commonly used are the Fourier series, which may be expressed as a summation of sines and cosines of harmonic frequencies. Fourier series may be used to determine the steady-state sinusoidal relating function if the input and output variations contain only frequencies which are integer multiples of a first-order frequency. Since the relating function is unique the Fourier transformation is not necessary if Fourier series representation is used. Although in theory the use of a Fourier series to determine the relating function would result in an exact solution, in practice only an approximation is obtained, since the series contains an infinite number of terms. Extensive treatments of the mathematical properties of Fourier series are given in many texts (14, 15, 16).

When the steady-state sinusoidal relating function is known and it is desired to obtain the response of the system to an input known only as a function of time the inverse transformation must be used. This procedure is defined as the inverse Fourier transform. With the usual assumptions made in engineering practice the approximation methods for obtaining the inverse transform closely parallel those for obtaining the direct transform.

The procedure for transforming many functions is laborious and time-consuming unless some means of mechanizing the process can be found. Machines for this purpose have been constructed (8, 17) and have been successfully used. They do, however, possess several basic shortcomings. Devices which approximate the function to be transformed by a series of transformable functions have proved to possess limited

accuracy at higher frequency ranges. Harmonic analyzers are not capable of producing continuous frequency functions. Many, if not all, of both types of machines are expensive.

An investigation has been made into a proposed method of obtaining an approximation of the Fourier transform or inverse Fourier transform. The device for performing this belongs to the first class of machines mentioned, i.e., those machines which perform the transformation of a series of functions. The term "torsional Fourier transformer" has been applied to the device, since the principal element is a torsional spring. Advantages of the torsional transformer appear to be continuous frequency information with greater accuracies and at lower costs than machines now in use.

CHAPTER 2

THE FOURIER TRANSFORM

2.1 Definition of a Fourier transform

The Fourier transform, denoted by the symbol [FT], is defined in its most general form by the equation

$$[FT]\,F(x) = \int_{-\infty}^{\infty} F(x)\, e^{-jzx}\, dx = G(z)$$

in which j represents the complex quantity $\sqrt{-1}$. In practice the equation usually indicates a transformation from the time domain to the frequency domain. Hence the relationship becomes

$$[FT]\,F(t) = \int_{-\infty}^{\infty} F(t)\, e^{-jwt}\, dt = G(w).$$

It should be noted that certain restrictions exist on the original function, F(t). If $\int_{-\infty}^{\infty} |F(t)|\, dt$ is finite, and F(t) is continuous, then G(W) exists. The function is usually limited in engineering practice to include only functions of the form

$$F(t) = 0 \qquad t < 0$$

$$F(t) \neq 0 \qquad 0 < t < T$$

$$F(t) = 0 \qquad t > T$$

The transformation in this special case has the form:

$$[FT]F(t) = \int_0^T F(t)e^{-jwt}dt$$

These restrictictions, although limited mathematically, include most functions encountered in engineering practice and simplify the problem of mechanizing the equation.

2.2 The Inverse Transform

The inverse transformation, from the frequency domain to the time domain, is defined as the inverse Fourier transform, represented by the symbol $[FT]^{-1}$, and defined as

$$[FT]^{-1}G(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(w)e^{-jwt}dw = F(t)$$

Additional conditions which will simplify the equation for the inverse transform may be found in Chapter 4.

This thesis is a description of the design and analysis of a device capable of obtaining either the Fourier transform or the inverse Fourier transform of a function which meets the conditions outlined above.

No attempt is made in this thesis to define the general properties of a Fourier transformation, nor the techniques involved in the solution of the transform equation for analytic functions. These features have been extensively covered in the literature (12, 14, 15). It should be pointed out, however, for those who may be more familiar with the LaPlace transformation, that the Fourier forms are special cases of the more general LaPlace integrals, in which the parameter jw replaces the complex parameter $s = \alpha + jw$.

2.3 The Relating Function

Potential applications of Fourier transforms for engineering purposes have been increased in recent years with the introduction of pulse testing of complete physical systems. The relating function concepts of Draper and his associates (1) utilize the Fourier transform as one of the major tools of analysis and design of linear systems.

The general form of the relating function is defined by the governing equation of the system

$$q_{out} = [RF]\, q_{in}$$

A special case of the general relating function can be defined as the quotient of the Fourier transform of the output quantity divided by the Fourier transform of the input quantity. This special case will be the only case considered in this discussion and will be referred to by the more general term "relating function". From the definition of the Fourier transform it is seen that this form of relating function is a function of frequency; or, as it is often used, a function of a non-dimensional frequency ratio.

For any physical system which may be associated with a linear differential equation this steady-state sinusoidal relating function is an operational form of the differential equation relating the input or independent variable and the output or dependent variable. The relating function, however obtained, is unique and can be used to obtain the steady-state response of the system to any sinusoidally varying input. Since the theory of Fourier series enables any arbitrary input to be expressed in terms of component sinusoidal frequencies, the relating function then becomes the only function necessary to express the complete behavior of a linear system.

2.4 Applications of the Relating Function

The implications of this concept have been exploited by Draper, McKay and Lees (1), Seamans and others (9, 18), particularly in the field of performance function analysis. This particular phase of instrument engineering deals with the problem of determining the relating function of an operating component or system for which the response has been determined for a known input. If desired it is even possible to obtain from the relating function, determined only as a frequency dependent plot, the linear integro-differential equation which most nearly represents the component or system.

Particular application of pulse methods of testing, together with the associated performance function analysis, have been made to the field of aircraft stability and control. (9, 18) A pulse input to the aircraft is applied in the form of a control surface movement. Recordings of both the input pulse and the resulting response of the aircraft are used to determine the frequency characteristics. The technique has been extended to include the determination of the actual stability derivatives of the aircraft. Results have compared favorably with those achieved as a result of more expensive and time-consuming sinusoidal excitation tests. In addition, tests can be performed by the pulse testing method under conditions of turn, bank, climb, or glide where the sinusoidal excitation technique cannot be used.

2.5 Other Applications of Fourier Transforms

Although results have not been published it is known that pulse testing methods have been used in evaluating the performance of heat exchangers, with results comparable to those obtained by using a sinusoidal input. The application of the pulse method to any rate process involving linear systems should be straightforward.

Fourier transforms of arbitrary functions are also employed in seismic research (17). The input pulse consists of a dynamite charge; the output is the recorded trace of the earth's vibrations. Since the vibrations in which interest is centered lie within a definite frequency spectrum Fourier transformation of the time dependent trace is used to provide the desired frequency information. The variable area Fourier analyzer described in Chapter 3 was developed for this purpose.

CHAPTER 3

MACHINE METHODS OF OBTAINING FOURIER TRANSFORMS

Computers capable of yielding the transform of an arbitrary function fall into two broad classifications, analogue and digital. Although the instrumentation differs widely within the analogue group, the governing equations for the machines follow one of two basic forms. An analogue transformer of each type will be discussed in some detail. Discussion of a torsional type transformer will be taken up in Chapter 4.

3.1 The Transient Analyzer

The first of the analogue devices is the Transient Analyzer developed in the Instrumentation Laboratory at M.I.T. (8). The analyzer is an electro-mechanical machine incorporating 24 resolvers as the principal elements. These resolvers are used as variable transformers. The rotors of these resolvers are connected to each other by gearing arranged in such a manner that the gear ratio between successive pairs of rotors is increased by unity. Thus the effective gear ratio between the first and last rotors is 24:1.

This analyzer can be used to obtain the approximate Fourier transform of an arbitrary function in the time domain. In general the function must satisfy the conditions as outlined in Chapter 2, i.e., it must be a pulse of finite time duration and pulse strength. The period of the function is first divided into 24 equal time increments and the

amplitude of the function at some average value in each increment is determined. This results in the function effectively being replaced by a series of step functions of varying amplitude.

A voltage proportional to the amplitude of the step function for the first increment of time is applied to the rotor coils of the first resolver. Similar voltages, proportional to the amplitudes of succeeding step functions, are applied to the respective rotors. Initially the rotors are aligned such that within each resolver the rotor coils are parallel to one of the two stator coils. The summation of the voltages from this set of stator coils would then be proportional to the integral of the step-function approximation. The second set of stator coils, which are arranged in space quadrature to the first set, would have no voltage output.

Rotation of the geared rotors by a fixed amount would cause any one of the first set of stator coils to have a voltage output equal to the original value multiplied by the cosine of the angle through which the particular rotor had turned. Similarly, the other stator coil for that resolver would have a voltage output equal to the original value multiplied by the sine of the angle of rotor rotation.

That the process results in an approximate Fourier transform can be demonstrated. Define $F(i\Delta t)$ as the step-function approximation to $F(t)$. Then, since rotation through an angle can be expressed mathematically as multiplication by the kernel $e^{-jw\Delta t}$, one approximation to the Fourier transform is given by

$$[FT]_{approx} F(t) = \sum_{i=1}^{24} \left[F(i\Delta t) e^{-jwi\Delta t} \right] \Delta t$$

Application of DeMoivre's theorem gives the expression

$$[FT]_{approx} F(t) = \sum_{i=1}^{24} F(i\Delta t)[\cos(wi\Delta t) - j \sin(wi\Delta t)]\Delta t$$

Thus the output of the transient analyzer consists of voltages proportional to the two summations.

The error arising from the approximation can be divided into two parts. The major error is due to the step-wise, rather than continuous, rotation of the function. Errors due to the step function approximation of the original function appear to be minor in comparison. Further discussion of the magnitudes of the errors will be found in Appendix B. For the present it is sufficient to note that this error will decrease as the number of time increments increases. Increasing this number, however, requires the addition of more resolvers; a rather expensive matter. Further complications arise in designing the required gear trains for the larger number of resolvers.

An example of results obtained by the use of the Transient Analyzer is shown in Figs. 1 and 2.

3.2 A Variable Area Type Fourier Analyzer

Another approach to the mechanization of the Fourier transform is reported by McDonal (17), as applied to problems in seismic research. This device was developed for use directly with variable area films obtained from a special seismic recorder, although its use with any type of variable area film is possible.

The function, F(t), is converted from a recorder area to a voltage by means of a light beam passing through the moving film onto a photocell. The amplified photocell voltage is fed into two potentiometers whose shafts are turned through angles proportional to cos wt and sin wt, respectively. This harmonic rotational motion is obtained by

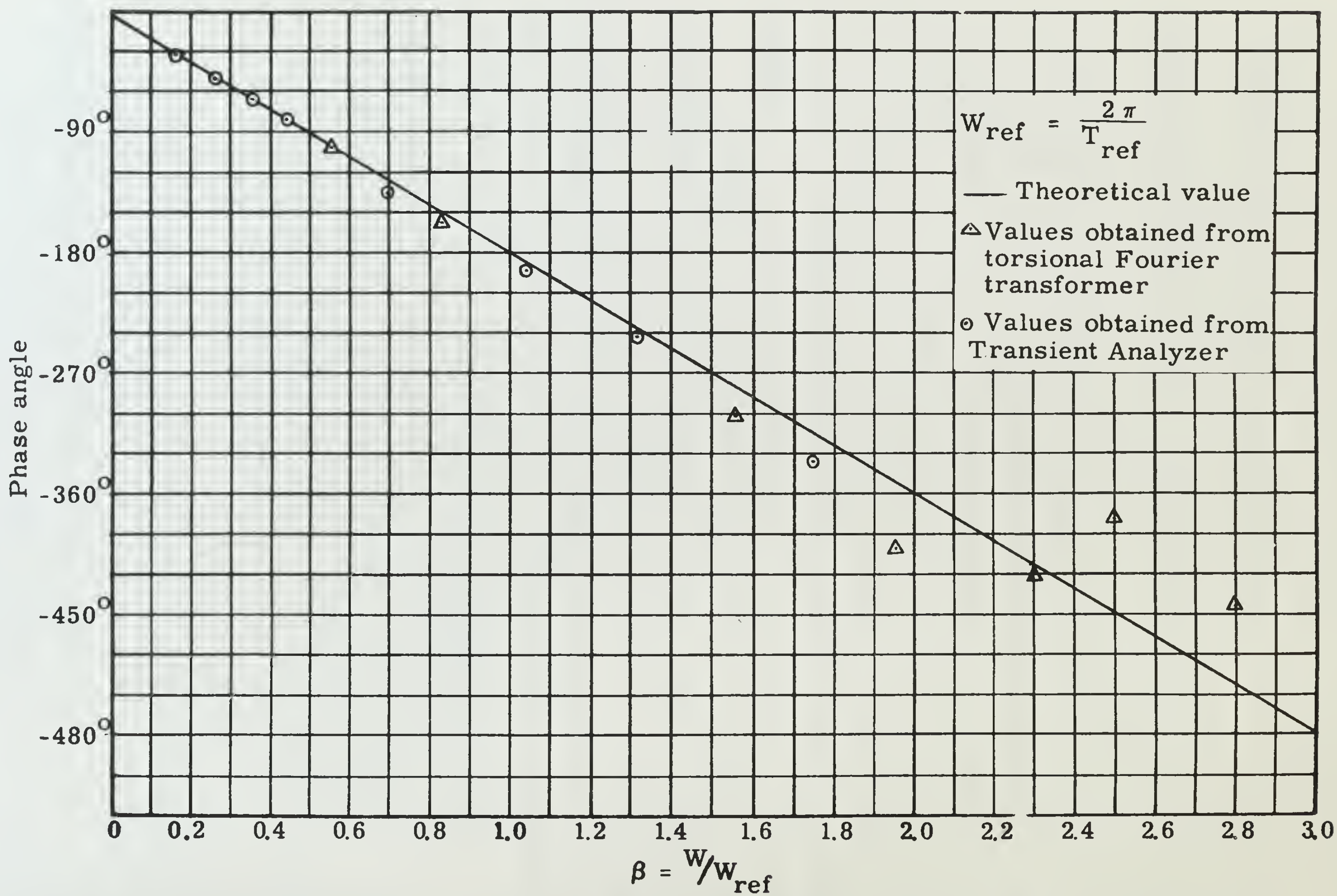

Fig. 1 Phase angle function of Fourier transform of displaced cosine pulse function.

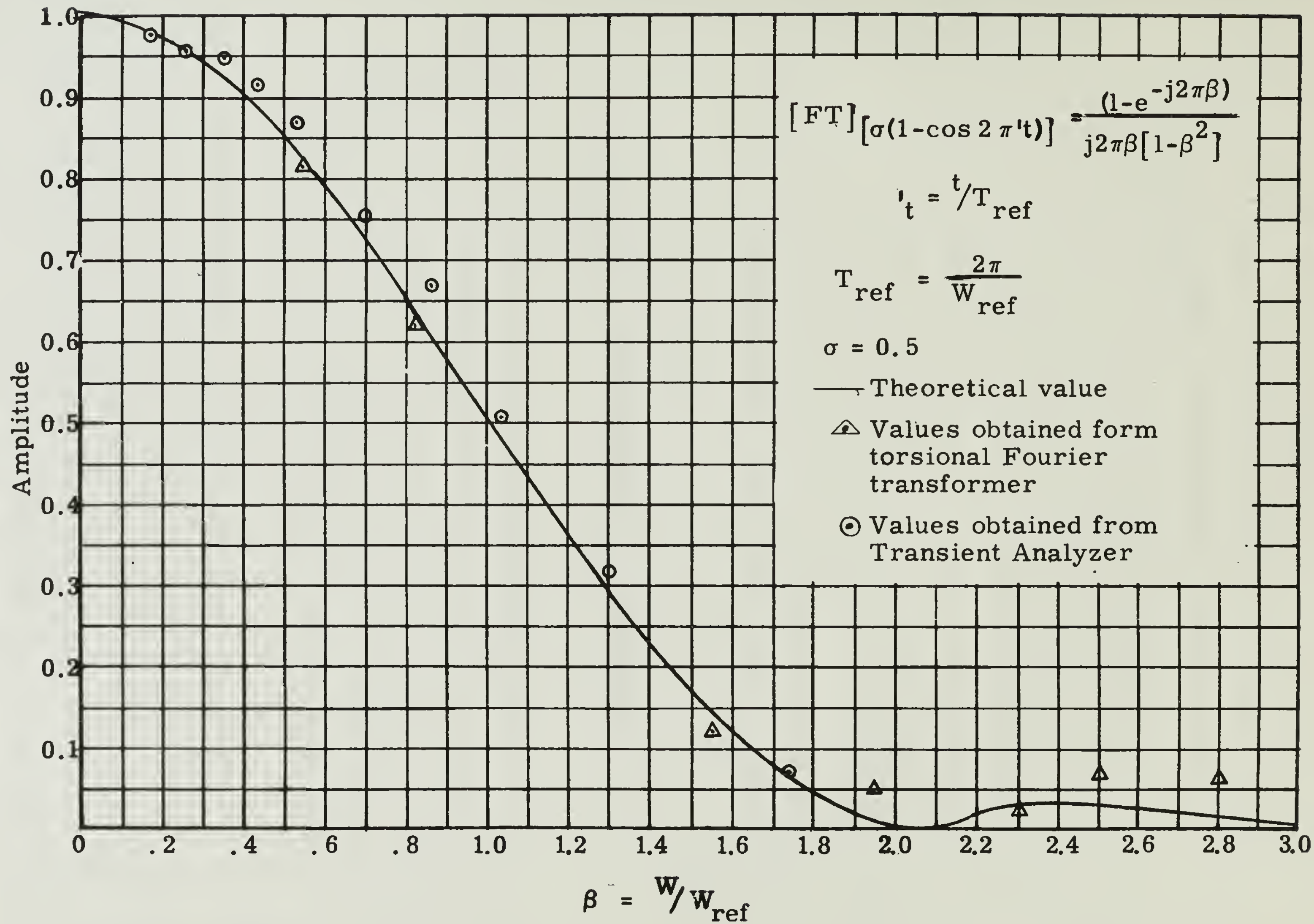

Fig. 2 Magnitude function of Fourier transform of displaced cosine pulse function.

an arrangement of Scotch yokes and pulleys, driven through gear boxes by the same constant speed motor which drives the variable area film.

Thus at any time the outputs of the two potentiometers are quantities proportional to the products F(t) sin wt and F(t) cos wt, respectively. Integration is accomplished by means of two velocity type servos utilizing servo motors and tachometric feedback. The outputs, expressed as shaft rotations, are proportional to the two components of the Fourier transform for a particular value of the variable w. For every additional frequency for which it is desired to evaluate the Fourier transform it is necessary to change the gear ratios between the driving motor and the Scotch yokes. As constructed the device is capable of a frequency range of 150 to 1.

A fundamental difference exists between the Transient Analyzer and the variable area type transformer. The governing equation for the latter is

$$[FT]F(t) = \int_0^T F(t)[\cos(wt) - j\sin(wt)]dt$$

This is seen to be the exact equation for the Fourier transformation, and hence no theoretical error would exist. However, the Transient Analyzer can be made to yield the approximate frequency function directly and continuously, while the variable area type transformer would produce only one point of the frequency function for each gear ratio used.

It should be noted that the variable area device has the property of producing the Fourier series function corresponding to the original time-dependent function. If successive integral values of gear ratios are used, successive transformations of a function will result in a harmonic analysis of that function.

Published reports indicate that a high degree of accuracy can be obtained with a device of this type. The disadvantages appear to be high initial cost and the labor required to cover a complete frequency spectrum.

3.3 Digital Computers

Since the physical details of digital computers vary widely no attempt will be made to describe an actual machine. The equations governing the process of obtaining a Fourier transform are the same for all digital computers

$$[FT]_{approx} F(t) = \sum_{i=1}^{n} F(i\Delta t)[\cos(wi\Delta t) - j\sin(wi\Delta t)]\Delta t$$

Since this is also the governing equation for the Transient Analyzer, the theoretical erros are also the same. However, a decreasing of the time interval, Δt, would not require an increase in equipment for the digital computer. Hence, any desired degree of accuracy can be attained.

CHAPTER 4

TORSIONAL REPRESENTATION OF A FOURIER TRANSFORM

4.1 Torsional Analogy

The basis for the torsional analogue of a Fourier transform is the analogy between the physical angle of twist of a torsion spring and the kernel, e^{-jwt}, which appears in the transform equation. Mathematically, the kernel indicates that a unit length is to be rotated through a variable angle in the complex plane. It is necessary to relate the physical variables of the spring to the mathematical variables, w and t.

To relate these variables, consider the effect of twisting a long shaft of uniform cross section, holding one end stationary. If restraint is not placed on the shaft to retain the original length the stresses in the shaft have been shown (19) to be pure shear. Furthermore, since the shaft is of uniform cross section, the stress distribution along the axis of the shaft will be uniform. This indicates that the angle of twist per unit length will be constant.

Thus the angular displacement of any cross section along the shaft is a linear function of two variables the angle through which the free end of the shaft has been twisted and the linear distance from the cross section to the fixed end. By relating the variable w to the angle of twist, and the variable t to the linear distance the analogy of cross section angular displacement to the kernel, e^{-jwt}, becomes evident.

The integrand appearing in the Fourier transform, expression is the product of a function, F(t), and the kernel e^{-jwt}. Therefore, it is necessary to modify the cross section of the shaft in a manner analagous to the variation of F(t). This however, must be accomplished without changing the linearity of twist per unit length of the shaft. To do this exactly would require the addition of material which would impose no additional resistance to shear in a plane perpendicular to the axis of the shaft, but which possesses rigidity in all planes containing the shaft axis.

This condition can be approximated by adding narrow strips of any rigid material mounted on the shaft with their long dimensions perpendicular to the shaft axis. Provided that the strips are completely separated from each other the condition that there be no restraint in shear is approximately true. The degree to which the approximation is satisfactory is dependent upon the width of the individual strips.

Again considering the shaft axis as the axis of the running variable, t, the lengths of the strips can be considered as representing the function F(t). Figure 3 illustrates such a representation in which the function is a displaced cosine pulse. Since F(t) is restricted to a real function the function displayed in Fig. 3 must lie in a real plane. Therefore, the plane perpendicular to the real plane which contains the shaft axis must represent an imaginary plane. This is the familiar "contour" representation of conformal transformation theory.

Rotation of the shaft will result in a presentation similar to that of Fig. 4. Projected areas will be seen in both the real and imaginary planes. It is the integration of these areas which will result in the two components of the desired frequency function. The governing equation for the process is

$$[FT]_{approx} F(t) = \sum_{i=1}^{n} F(t)[\cos(wi\Delta t) - j\sin(wi\Delta t)]\Delta t$$

Fig. 3 Projection of displaced cosine pulse function on real plane (W = 0).

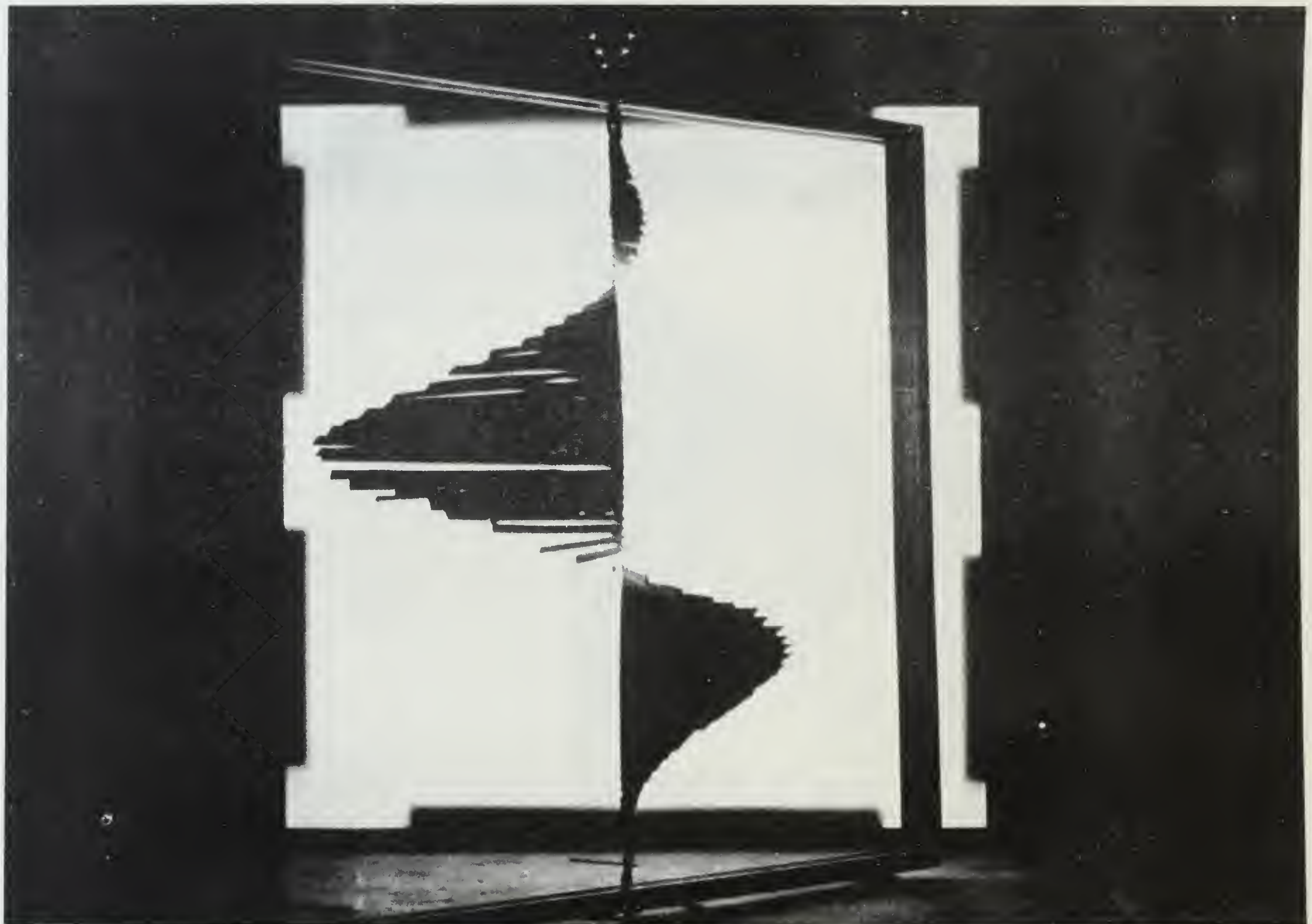

Fig. 4 Projection of displaced cosine pulse function on imaginary plane ($W \neq 0$).

This approximation differs from that for the transient analyzer in that the actual function F(t) is used in the torsional representation instead of a step-function approximation. Qualitatively this will reduce the error arising from the approximation, although a quantitative analysis as given in Appendix B indicates that the major source of theoretical error is the same in the two systems.

4.2 Inverse Fourier Transforms

As defined in Chapter 2 the inverse Fourier transform is found by evaluating an integral of a function in the frequency domain.

$$[FT]^{-1} G(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(w) e^{jwt} dw = F(t)$$

In general it can be assumed that G(w) will be a complex function with both magnitude and phase angle functions. In engineering practice functions most frequently encountered are capable of undergoing further simplification. Conditions on the function specify that the magnitude of the function G(w) be an even function with respect to positive and negative values of w, and that the argument or phase angle be an odd function. With the further stipulation that the function does not exist for negative values of t, it can be shown (1) that the reduced form of the equation becomes

$$[FT]^{-1} G(w) = \frac{2}{\pi} \int_{0}^{\infty} Re\,[G(w) \cos(wt)]\, dw$$

where Re $[G(w)]$ is the real part of the function G(w).

The problem of obtaining the inverse Fourier transform by a torsional device is similar to that for the direct transform. The function is plotted, either in terms of w or a nondimensional frequency ratio. Division of the function into increments of width Δw or $\Delta\beta$ is made, and the function placed on the torsional shaft. Various values of t are

introduced by means of shaft rotation, with integration of projected areas being accomplished as for the direct transform. Note that the process involves only integration along the real plane, since the conditions specified for the function G(w) are equivalent to specifying that the inverse transform of G(w) be a real, and not a complex, function of t.

The complete similarity of form which exists between the direct and inverse transforms indicates that, by the interchange of physical variables, any device capable of performing the direct transformation is also capable of yielding the inverse transform. Because of this similarity an analysis of the inverse transformation process is not included in the descriptions of the transient analyzer and the variable area type analyzer.

CHAPTER 5

EXPERIMENTAL INVESTIGATION OF A TORSIONAL FOURIER TRANSFORMER

5.1 Purpose of the Investigation

The purpose of this investigation was to determine the possibilities and limitations of obtaining the Fourier transform of an arbitrary function by means of a torsional analogue. A description of the equipment used to perform this investigation may be found in Appendix A.

5.2 Selection of Functions

Functions whose Fourier transforms can be calculated directly were chosen to test the performance of the torsional Fourier transformer. To carry out the relating function concept functions representing a possible input to a physical system and the response of a known system to this same input were transformed.

To this end a displaced cosine pulse was chosen to represent the input to a second order system. (Fig. 3). This particular pulse shape is particularly adapted to actual testing of physical systems since at the beginning of the pulse both the value of the function and its derivative are zero. The rectangular pulse function, on the other hand, is discontinuous in value at the initial time as well as possessing a first derivative of infinite magnitude. As a result it is impossible to

build testing equipment for many physical systems which will produce a true rectangular pulse function input. The displaced cosine pulse is relatively easy to generate and to apply to most physical systems.

5.3 Description of Investigation

A plot of the function to be transformed was made on corrugated paper which was then cut into strips of equal width. These strips were mounted on a flat spring, one end of which was clamped securely to a mounting frame. The other end of the spring was held by a slotted shaft which could be rotated with respect to the frame. The entire device was placed in front of a lighted opalescent screen.

Initially the frame was so aligned that the plane of the corrugated strips was perpendicular to the line of sight between the frame and a camera mounted some distance away. Precautions were taken to mount the camera on the perpendicular bisector of the spring axis at a sufficient distance to minimize the effects of parallax. The first picture was taken with the spring untwisted, representing a zero value of frequency. (Fig. 3). After an angle of twist had been introduced into the spring pictures were taken both with the frame in its initial position and after being rotated 90°. The process was repeated for additional angles of twist. Figure 4 illustrates a typical result for an intermediate angle of twist.

5.4 Evaluation of Data

Enlargements of each picture were made to a standard size, with a constant enlarger setting. The occluded areas of the enlarged pictures represent the desired components of the transformed function. Integration of these areas, with due regard to algebraic sign, was accomplished by cutting out the outline of the projected areas and weighing the resultant pieces of photographic paper on an analytic balance.

The process was readily adaptable to nondimensionalization. The reference value of magnitude was taken as the weight of the area on the real plane at zero angle of twist. The relationship between the angle of twist, θ , and the nondimensional frequency ratio, β , for a function of period T, is

$$\theta = n W \Delta t = W T = W\, T_{ref} \frac{T}{T_{ref}}$$

since

$$W_{ref} = \frac{2\pi}{T_{ref}}$$

then

$$\theta = \frac{2\pi W}{W_{ref}} \frac{T}{T_{ref}} = 2\pi\beta \frac{T}{T_{ref}}$$

For the results presented in this thesis the reference time was taken as the undamped natural period of a second order system. The nondimensional frequency ratio, β , then becomes the ratio of the forcing frequency, w, to the undamped natural frequency, w_n. It should be noted that this choice of reference time has no significance for the transform of a function such as a displaced cosine pulse function unless the ratio of the pulse duration to the undamped natural period is specified.

5.5 Results of the Investigation

Results of the investigation are presented in Figs. 5-9. The magnitude functions are plotted on logarithmic coordinates, the phase angle functions on a semi-logarithmic basis. Results of the transformation of the displaced cosine pulse function are also shown on a non-logarithmic plot. (Figs. 1 and 2)

Included on the figures for comparison are the calculated curves of the magnitude and phase angle functions of the Fourier

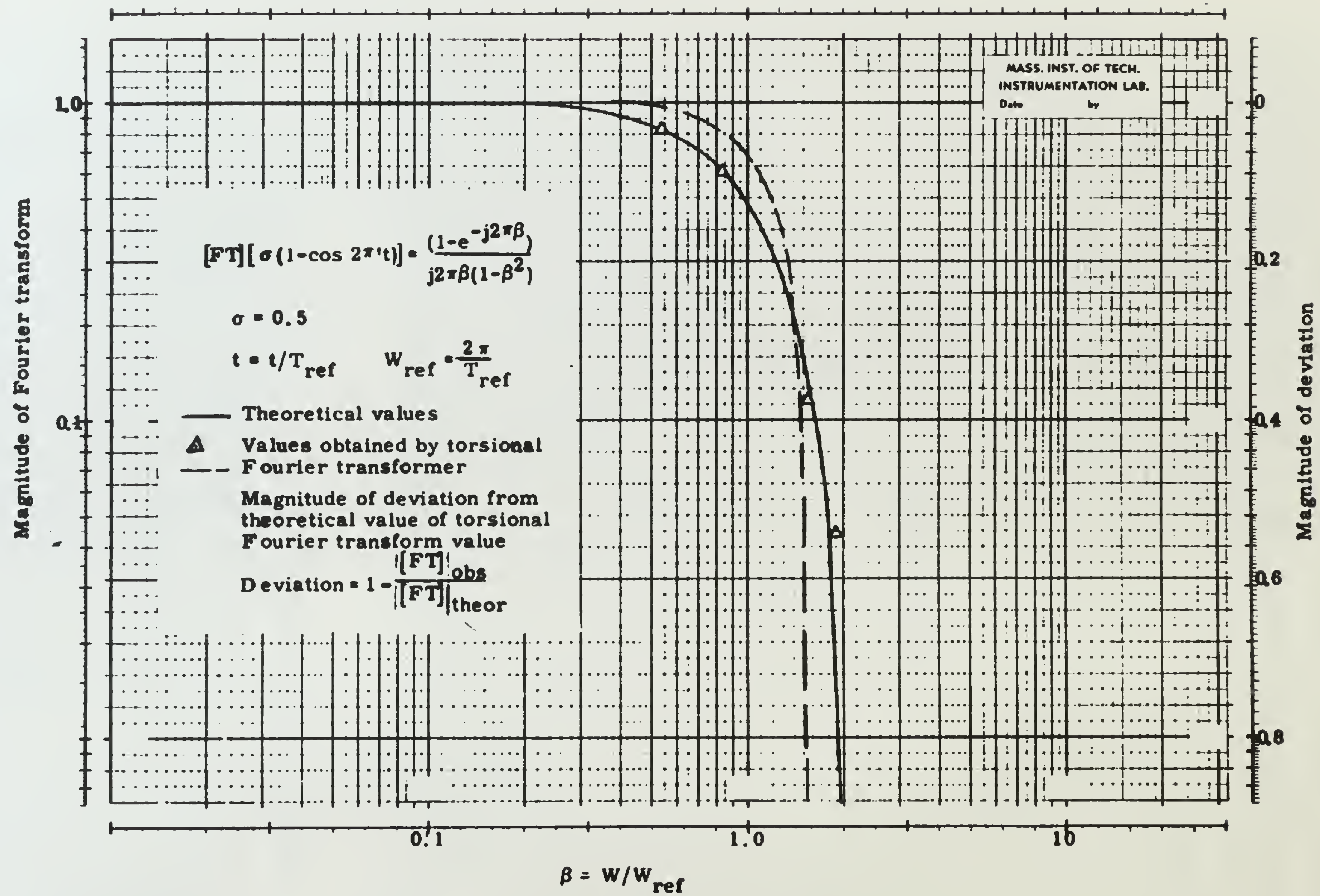

Fig. 5 Logarithmic plot of magnitude function of the Fourier transform of a displaced cosine pulse function.

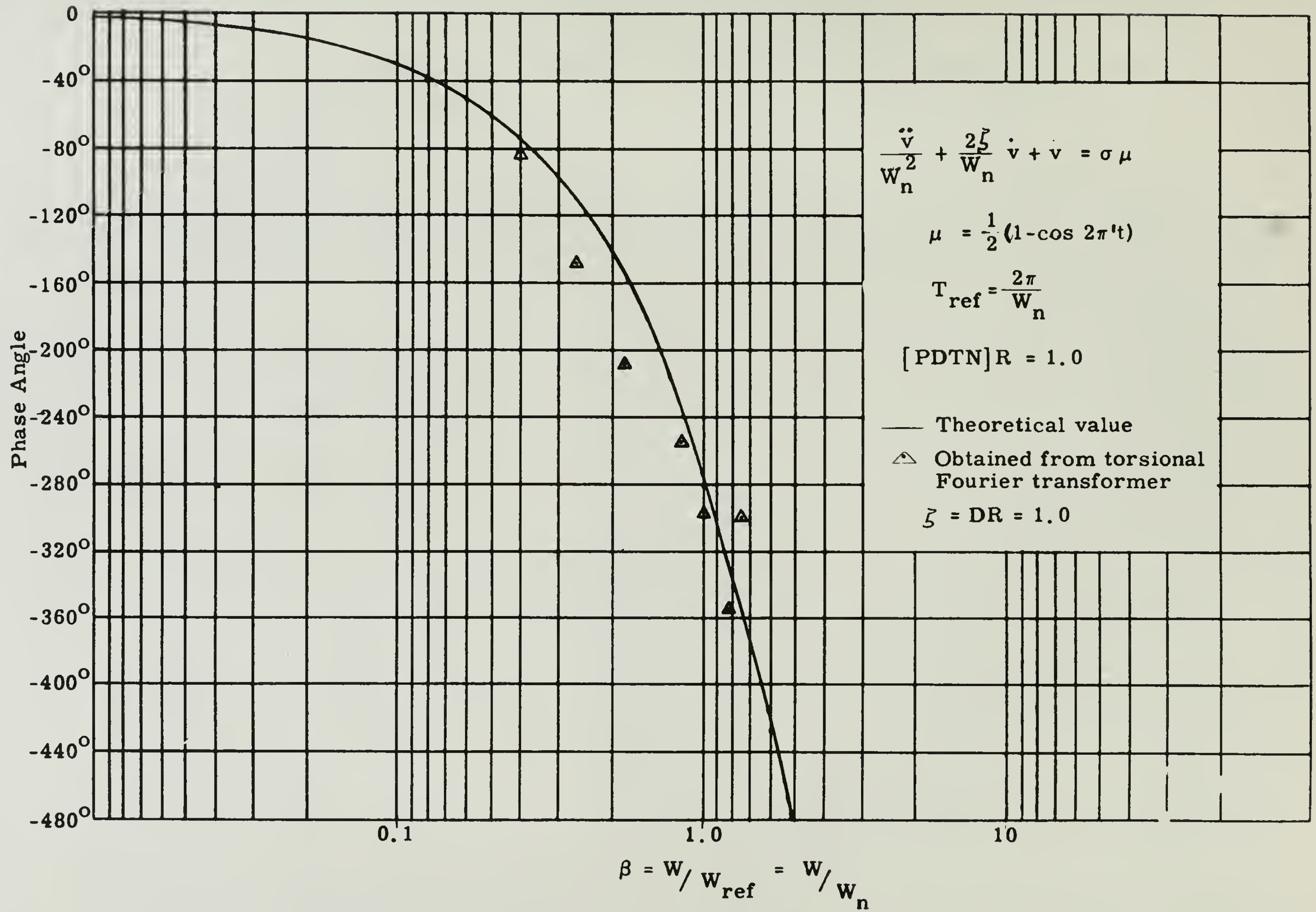

Fig. 6 Phase angle function of Fourier transform of the response of a second-order system (DR=1.0) to a displaced cosine pulse function input.

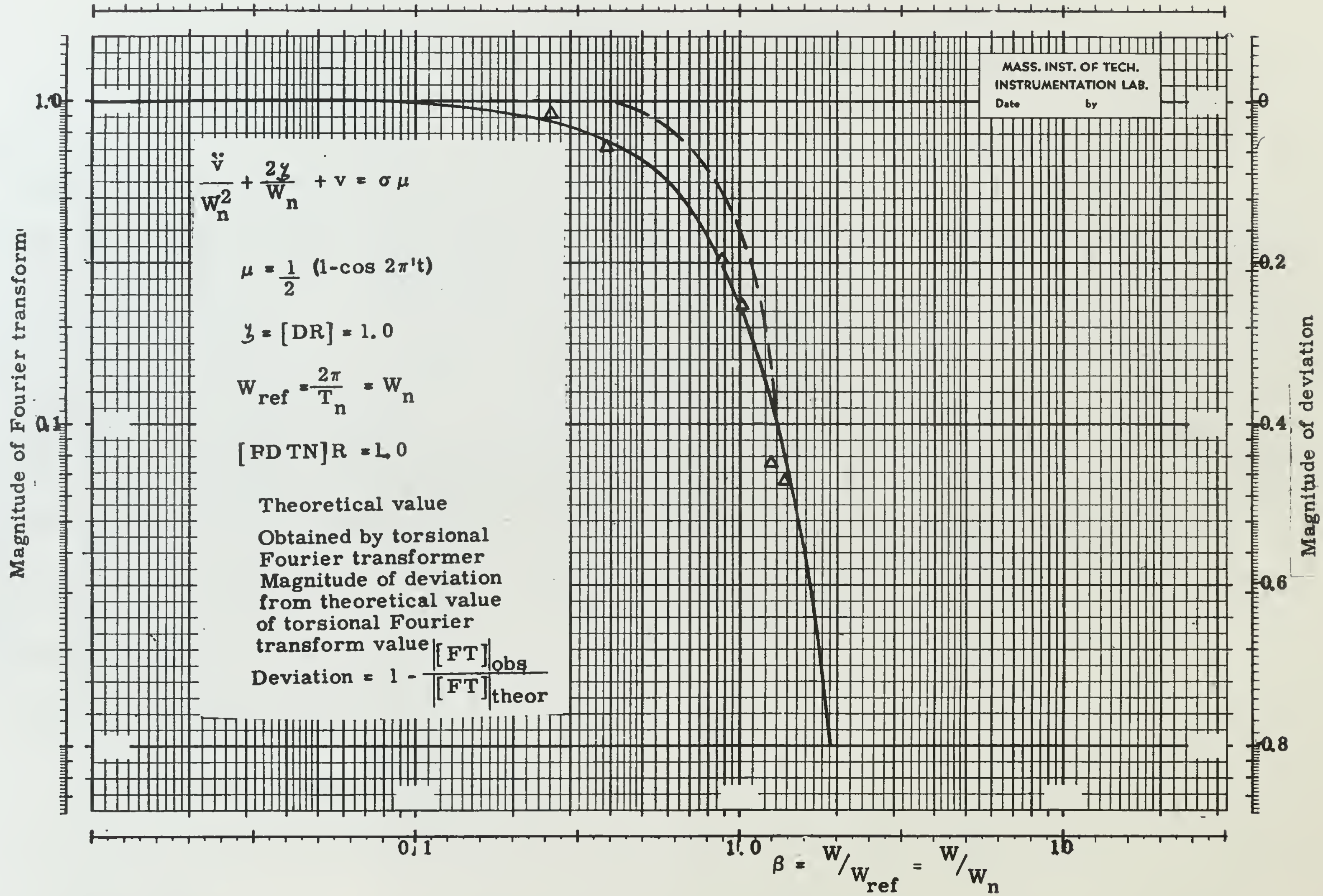

Fig. 7 Magnitude function of the Fourier transform of the response of a second-order system (DR=1.0) to a displaced cosine pulse input function.

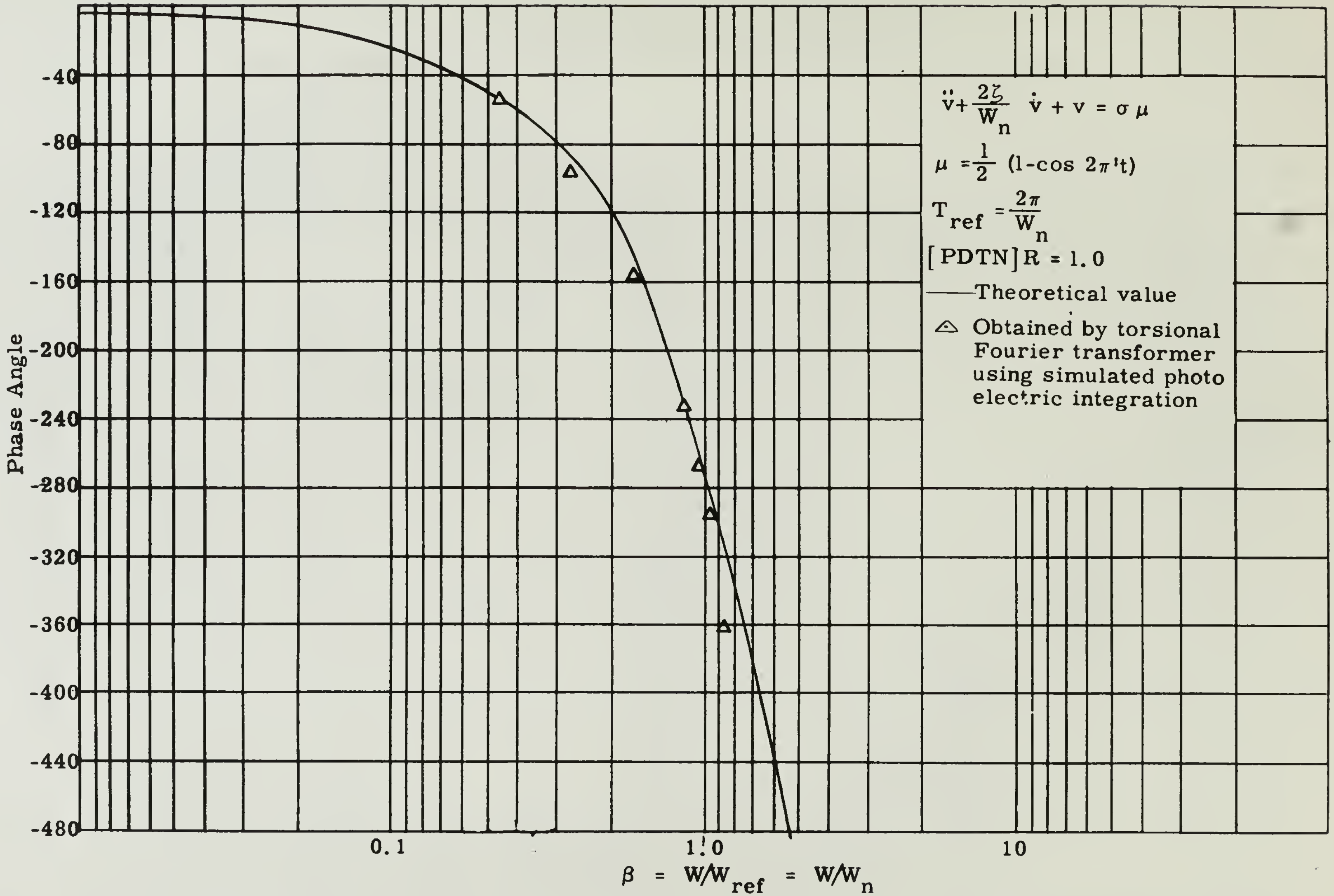

Fig. 8 Phase angle function of Fourier transform of the response of a second-order system (DR=0.5) to a displaced cosine pulse function input.

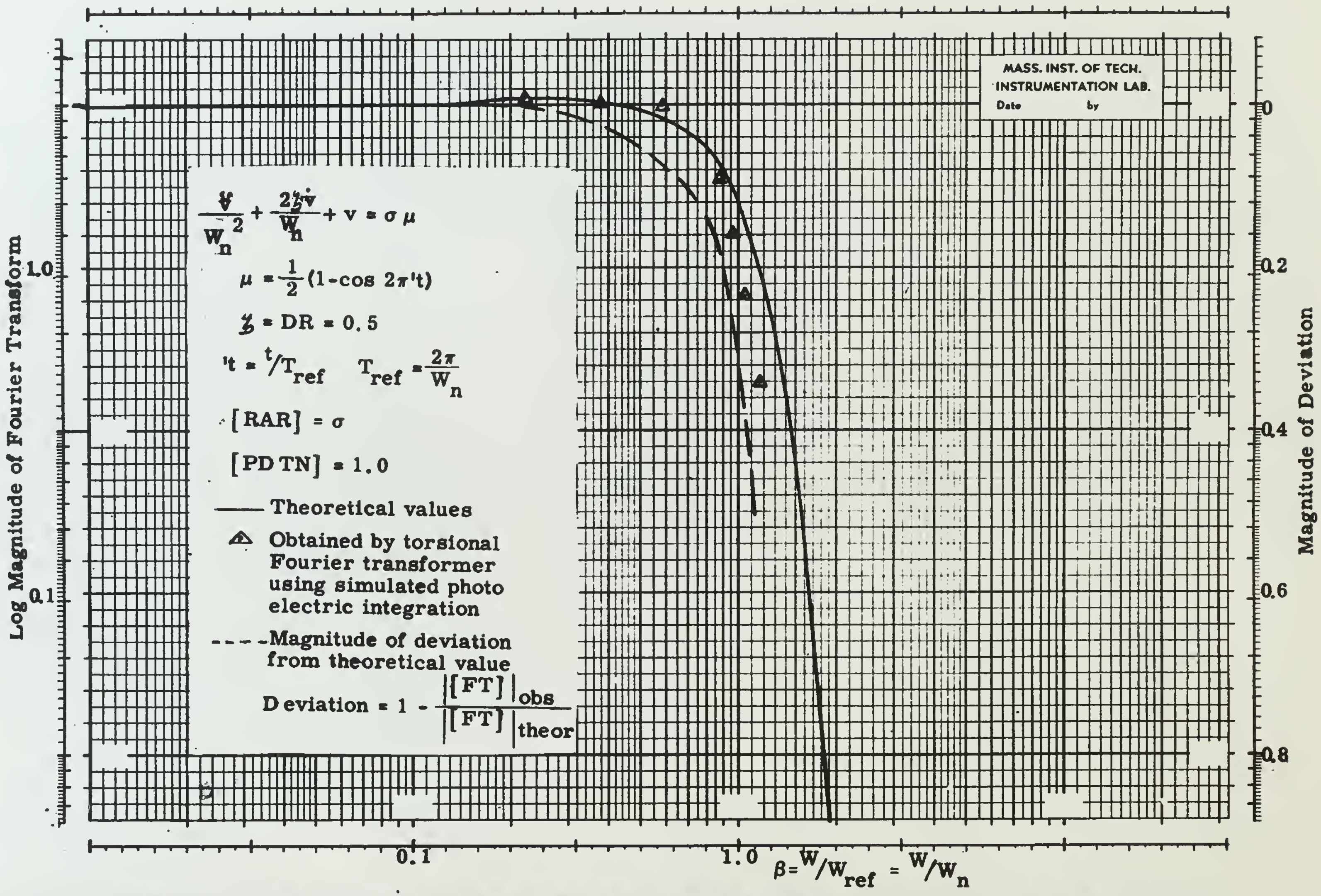

Fig. 9 Magnitude function of the Fourier transform of the response of a second-order system (DR=0.5) to a displaced cosine pulse input function

transforms. The deviation from the calculated values of the observed results are also presented. The transform of the displaced cosine pulse function was obtained from a table of transform pairs (1), then reduced by a process of nondimensionalization to the form

$$[FT]\left[\frac{1}{2}(1-\cos \omega' t')\right] = \frac{1-e^{-j2\pi\beta}}{j2\pi\beta(1-\beta^2)}$$

The response curves of the second order system were obtained from analogue computer data (1). The theoretical Fourier transforms of the response functions were obtained by use of the relating function concept, employing the relating functions for second order systems to be found in Ref. 1 .

$$[FT]q_{out} = [FT]q_{in}[RF]_{system}$$

Since both the relating function and the Fourier transform of the input function were available in logarithmic form, only an additive process was necessary to obtain both he magnitude and phase angle response functions.

Figures 8 and 9 require additional explanation. The entire experimental investigation was carried out using the photographic method previously described for the integration of areas. However, it was conceived that if a practical instrument were to be built, integration would probably be dependent upon the use of an integrating photo-electric system. As a study of Fig. 4 will show, and as discussed in Appendix C, there are gaps between adjacent strips as seen in the projected plane. These gaps do not materially affect the envelope of the areas, but would presumably affect the values obtained by photo-electric integration. The curves of Figs. 8 and 9, therefore, are the results of eliminating these gaps from the weighed areas.

A derivation of the errors inherent in an approximation of the Fourier transform by a step-wise process such as the one used in the investigation is given in Appendix B. A partial evaluation of the uncertainties present in this investigation may be found in Appendix C.

CHAPTER 6

UNCERTAINTIES IN A TORSIONAL FOURIER TRANSFORMER

6.1 Introduction

A discussion of the uncertainties existing in a representation of a Fourier transform by a torsional device must necessarily depend upon the details of the actual device used. Since the experimental investigation described in this thesis employed certain techniques which would not be used in a practical system the discussion of possible sources of uncertainty will not be limited to the experimental methods. A more detailed account of the uncertainties considered in the experimental investigation is given in Appendix C.

There are two phases to the torsional representation of a Fourier transform. The function first undergoes a step-wise rotation. Projected areas are then integrated to produce the desired components of the frequency function. Uncertainties exist in both the rotation and integration phases.

6.2 Rotational Uncertainties

The use of a torsion spring as the principal element introduces several sources of uncertainty. Nonuniformity of cross section or lack of homogeneity of the metal will result in variations in the angle per unit length. This effect can be minimized by maintenance of closer physical tolerances, but can never be completely eliminated.

Local differences in the degree of bonding achieved between the spring and the strips results in variations in the angular separation between adjacent strips. This effect may be decreased by use of a more suitable adhesive, or changing the design to allow a larger bonding surface on each strip. The use of a flat spring with strips fastened to only one side caused the strips, when rotated, to become tangent to a circle instead of being rotated about the spring axis. This uncertainty can be eliminated entirely by the use of a helical spring as the principal element. The strips would then be fastened in such a manner that their axes would intersect the axis of the spring.

6.3 Uncertainties of Integration

In the experimental investigation there were two major causes of integration errors. As shown in Fig. 4, twisting the spring results in a vertical separation occuring between adjacent strips. This is a direct result of using a spring with a rectangular cross-section. Substitution of a close coiled helical spring for the flat spring would eliminate most, if not all, of this error.

The photographic technique used in evaluating the experimental data was the source of the other integration error. As explained in Appendix A, a weighing method was used to obtain values proportional to the integrated areas. Uncertainties arose as a result of nonuniformity of paper density. These errors would be eliminated by the use of a photo-electric integration system. However, it is believed that the photo-electric system would possess uncertainties of comparable magnitude.

6.4 Summary

Uncertainties in the torsional transformer representation of a Fourier transform occur in terms of angular deviations and integration

deviations. By proper design some of the uncertainties present in the experimental investigation may be eliminated entirely. All of the uncertainties considered may be reduced in designing a more practical device.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

This thesis was undertaken to investigate the possibilities of a torsional Fourier transformer. Experimental investigation has shown that a torsional device will produce an approximation to the Fourier transform of certain functions. Sources of uncertainty which may be present in a torsional representation have been investigated, and design criteria established to reduce the level of these uncertainties.

The two fundamental analogue methods of obtaining a Fourier transform approximation have been compared. The basic differences have been shown to be

(1) The method exemplified by a harmonic analyzer will produce a transform with no theoretical error, but will not provide continuous frequency information.

(2) The approximation produced by a device such as a torsional Fourier transformer contains an inherent error, but is capable of being presented as a continuous frequency function.

Inverse transformations have been considered. The ability of a torsional transformer to perform the inverse transformation has been shown to be subject to the same limitations as exist for the direct transformation.

An approximation of the errors inherent in any device which obtains the Fourier transform by a step-wise process has been found for cases of special interest. The effect of increasing the number of steps or increments in the process has been demonstrated.

7.2 Recommendations

Based upon design criteria established in this thesis it is recommended that an effort be made to produce a torsional Fourier transformer of a practical nature. Specifically, the design should incorporate.

(1) a helical spring as the principal element of the torsional transformer,

(2) a photo-electric integration system,

(3) means for recombining the two component functions to obtain a single continuous frequency-dependent function.

Preliminary studies were conducted to determine a suitable photo-electric integration system. Results indicated that a system employing two light sources and one photo-cell should be simpler and more accurate than other configurations considered. The two light sources, for positive and negative area integration, respectively, would be alternately pulsed. This system would avoid the necessity for employing balanced photo-cell circuits or mechanical shutter arrangements.

It is suggested that further work be done to determine the inherent errors of the torsional analogue. The possibility exists that gain compensation might be applied to reduce the output error of a device such as the one suggested above. Such compensation could easily be introduced by the same control as used to introduce twist into the torsion spring.

APPENDIX A

DESCRIPTION OF THE EXPERIMENTAL TORSIONAL FOURIER TRANSFORMER

The experimental device used to conduct this investigation is shown in Fig. 10. There are two separate pieces of equipment, the light box and the mounting frame for the torsion spring. Figure 10 shows the apparatus in position for an initial photograph, with an arbitrary function arranged on the spring.

The lighting box consisted of a wooden frame, 21 in. square, containing five 25 watt flourescent light tubes, and fitted with an opalescent glass front. The sole function of the box was to provide an evenly lighted background for photographic integration.

Channel aluminum was used in the construction of the mounting frame, which measured 18 in. high by 20 in. wide. A steel slotted post was fitted in the middle of the bottom channel of the frame, with the slot arranged at a 45° angle to the frame. A tapered hole in the post at right angles to the slot provided a means of securing the end of the steel spring. In the middle of the top of the frame a brass slotted shaft was fitted in a bushing, with the outer end of the shaft connected by a set screw to a knob. The brass shaft was spring loaded to permit motion axially along the flat spring which stretched vertically across the middle of the frame. Slots in both the shaft and post were cut to a width of 0.008 in. to prevent movement of the spring within the slots.

The heart of the torsional device was the flat spring. Various sizes were tried, with the final choice being a strip of spring steel 1/4 in. wide by 0.006 in. thick. This appeared to offer a reasonable degree of stiffness while permitting sufficient twist to adequately cover the desired range of twist angles.

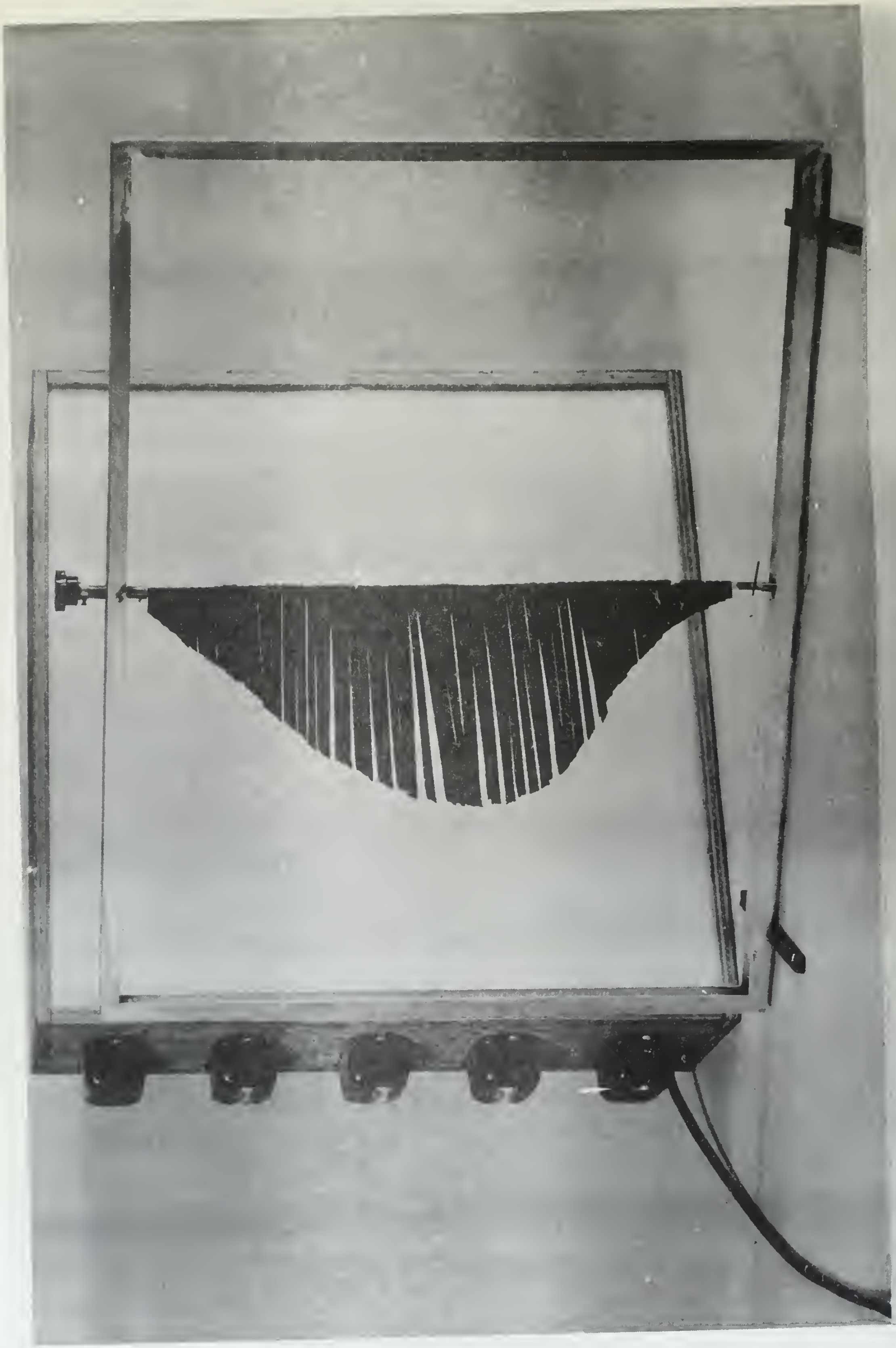

Fig. 10 Experimental equipment for investigation of a torsional Fourier transformer.

Finding a suitable material from which to prepare the function representation presented another problem. Advertising display paper, consisting of a corrugated layer bonded to a backing sheet, proved to possess sufficient rigidity when cut in thin strips. It was found from experience that best results were obtained by cutting a rectangular piece of paper into 1/4 in. strips which were left joined at one end. The spring was then fastened to the unjoined ends of the strips, and the function was cut to the desired shape. The use of a standard paper cutter is recommended for the initial cutting of the strips; the slightly jagged edge left by a razor blade or scissors results in interference between adjacent strips.

The most time-consuming problem encountered was the finding of a suitable method of securing the paper to the spring. Although the paper strips were light in weight, the bonding surface on each strip was only 1/16 sq. in. In addition to the weight effects, the twisting of the spring resulted in additional shearing forces being exerted on the bond. Of the many adhesives tested only one was found to be satisfactory; a double-sided pressure-sensitive tape produced by Minnesota Mining and Manufacturing Co. Best results were obtained by taking photographs immediately after fastening the paper to the spring.

It should be noted that it is necessary to mount the spring with its axis vertical, if gravity effects on the uniformity of twist are to be eliminated.

Photos were taken, as described in Chapter 5, with a 35 mm. camera mounted at a distance of eight feet from the spring. The camera was mounted on the perpendicular bisector of the spring axis, with the line of sight also perpendicular to the real plane of the function.

APPENDIX B

ERRORS INVOLVED IN APPROXIMATIONS TO A FOURIER TRANSFORM

The error involved in approximating a Fourier transform by a step-wise process such as is used in the torsional Fourier transformer is difficult to evaluate. Quantitative errors vary greatly with the degree of regularity or "smoothness" of the function transformed. Laning has shown (20) that the error calculation inevitably leads to an infinite series solution, the form of the series being dependent upon the form of the approximation of the original function.

Therefore, emphasis in this appendix is primarily upon evaluating the effect on the magnitude of the error function of increasing the number of increments of the running variable. Since the error function is of more general interest when expressed in non-dimensional terms, it is necessary to first define the non-dimensional form of the Fourier transform.

$$[FT]\, F(\acute{t}) = \int_{-\infty}^{\infty} F(\acute{t})\, e^{-j2\pi\beta\acute{t}}\, d\acute{t} \qquad (1)$$

where

$$\beta = \frac{W}{W_{ref}} \qquad \acute{t} = \frac{t}{T_{ref}} \qquad T_{ref} = \frac{2\pi}{W_{ref}}$$

This is exactly equivalent to the dimensional form of the Fourier transform as defined in Chapter I.

To evaluate the error involved in an approximation to the Fourier transform of an arbitrary function assume an approximation of the form

$$[FT]_{approx}F('t) = \sum_{i=-\infty}^{\infty} \int_{i\Delta't}^{(i+1)\Delta't} F('t)e^{-j2\pi\beta i\Delta't}d't \qquad (2)$$

which is the general case of the approximation inherent in the torsional Fourier transformer.

Define

$$(\mathcal{E})[FT]F('t) = [FT]F('t) - [FT]_{approx}F('t)$$

then

$$(\mathcal{E})[FT]F('t) = \int_{-\infty}^{\infty} F('t)e^{-j2\pi\beta 't}d't - \sum_{i=-\infty}^{\infty} \int_{i\Delta't}^{(i+1)\Delta't} F('t)e^{-j2\pi\beta i\Delta't}d't \qquad (3)$$

[FT]F('t) can be written as

$$[FT]F('t) = \sum_{i=-\infty}^{\infty} \int_{i\Delta't}^{(i+1)\Delta't} F('t)e^{-j2\pi\beta 't}d't \qquad (4)$$

substituting in (3) and collecting terms

$$(\mathcal{E})[FT]F('t) = \sum_{i=-\infty}^{\infty} \int_{i\Delta't}^{(i+1)\Delta't} F('t)(e^{-j2\pi\beta 't} - e^{-j2\pi\beta i\Delta't})d't \qquad (5)$$

Define a new variable

$$'\tau = 't - i\Delta't \quad \text{then} \quad d't = d'\tau$$

and

$$(\mathcal{E})[FT]F('t) = \sum_{i=-\infty}^{\infty} \int_{0}^{\Delta't} F('\tau + i\Delta't)(e^{-j2\pi\beta i\Delta't})(e^{-j2\pi\beta '\tau} - 1)d'\tau$$

$$(\mathcal{E})[FT]F('t) = \sum_{i=-\infty}^{\infty} e^{-j2\pi\beta i\Delta't} \int_{0}^{\Delta't} F('\tau + i\Delta't)(e^{-j2\pi\beta '\tau} - 1)d'\tau \qquad (6)$$

At this point it becomes necessary to make certain assumptions concerning the function, **F('t)**. One reasonable assumption is that

$$F('t) \approx F(i\Delta't) + '\tau \frac{d}{d't}[F(i\Delta't)]$$

within the interval $i\Delta't < 't < (i+1)\Delta't$.

Substitution into (6) yields

$$(\mathcal{E})[FT]F('t) = \sum_{i=-\infty}^{\infty} F(i\Delta't)e^{-j2\pi\beta i\Delta't}\int_0^{\Delta't}(e^{-j2\pi\beta'\tau}-1)\,d'\tau$$

$$+\sum_{i=-\infty}^{\infty}\frac{d}{d't}[F(i\Delta't)]\,e^{-j2\pi\beta i\Delta't}\int_0^{\Delta't}{'\tau}(e^{-j2\pi\beta'\tau}-1)\,d'\tau \qquad (7)$$

Performing the indicated integration

$$(\mathcal{E})[FT]F('t) = \sum_{i=-\infty}^{\infty} F(i\Delta't)e^{-j2\pi\beta i\Delta't}\left[\frac{(e^{-j2\pi\beta\Delta't}-1)}{-j2\pi\beta\Delta't}-1\right]\Delta't$$

$$+\sum_{i=-\infty}^{\infty}\frac{d}{d't}[F(i\Delta't)]\Delta't\,e^{-j2\pi\beta i\Delta't}\left[\frac{e^{-j2\pi\beta\Delta't}}{j2\pi\beta\Delta't}-\frac{1}{2}+\frac{(e^{-j2\pi\beta\Delta't}-1)}{(2\pi\beta\Delta't)^2}\right]\Delta't. \qquad (8)$$

Since

$$\frac{d}{d't}[F(i\Delta't)]\Delta't = F(i+1)\Delta't - F(i\Delta't)$$

eq. (8) can be expressed as

$$(\mathcal{E})[FT]F('t) = \left[\frac{(e^{-j2\pi\beta\Delta' t}-1)}{-j2\pi\beta\Delta' t} - 1\right] \sum_{i=-\infty}^{\infty} F(i\Delta' t)e^{-j2\pi\beta i\Delta' t}\Delta' t$$

$$+\left[\frac{e^{-j2\pi\beta\Delta' t}}{j2\pi\beta\Delta' t} - \frac{1}{2} + \frac{(e^{-j2\pi\beta\Delta' t}-1)}{(2\pi\beta\Delta' t)^2}\right] \sum_{i=-\infty}^{\infty} \{F[(i+1)]\Delta' t - F(i\Delta' t)\} e^{-j2\pi\beta i\Delta' t}\Delta' t.$$

In considering only the magnitude of the error function it is seen that the first summation can be expressed as the product of a deviation factor and a Fourier transform approximation. The second summation includes a term involving the difference between the value of F('t) at successive points. This latter term, from the conditions imposed upon F('t), must change sign during the period of the function. Furthermore, for functions which are symmetric, such as the displaced cosine pulse function, the effect on the magnitude of the error function disappears.

Therefore, an approximation to the magnitude of the error function becomes

$$\left|(\mathcal{E})[FT]F('t)\right| = \left|\frac{(e^{-j2\pi\beta\Delta' t}-1)}{-j2\pi\beta\Delta' t} - 1\right| \left|[FT]_{approx} F('t)\right|$$

This differs from the usual definition of a deviation function, which is defined as the ratio of the error to the true value. This can be expressed as

$$(D)[FT]F('t) = \frac{[FT]F('t) - [FT]_{approx}F('t)}{[FT]F('t)} = 1 - \frac{(-j2\pi\beta\Delta' t)}{(e^{-j2\pi\beta\Delta' t}-1)}.$$

This deviation function is plotted in Fig. 11 for two values of Δ't, together with observed deviation curves of an input function transform

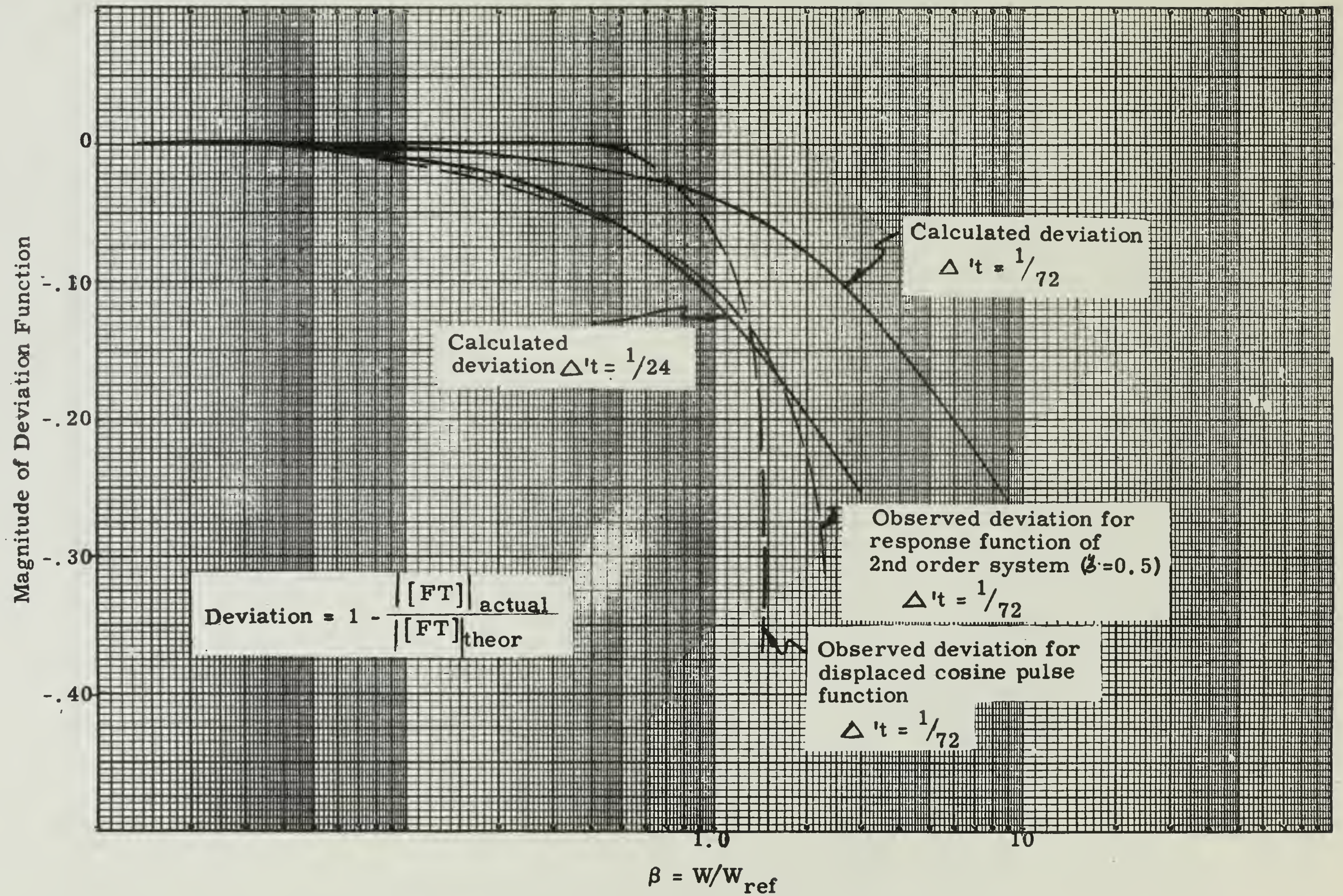

Fig. 11 Deviation functions of Fourier transform approximations.

and a response function transform. Several observations should be made, however, concerning a direct comparison between the observed and calculated curves.

First, T_{ref} for Fig. 11 is taken as the period of the function, hence the apparent disagreement between the deviation curve for the response function and the deviation curve of Fig. 9, in which T_{ref} was taken as the undamped natural period of the system. Secondly, the fact that the observed deviation is less than the theoretical deviation for the input function may arise from the method of obtaining the integrated areas for this case. The rapid increase in the observed deviation functions near $\beta = 2.0$ is to be expected since the theoretical value of the transforms at this point is zero.

No direct comparison with the transient analyzer results can be made, since the transformed function in this case was more nearly

$$\frac{1}{2}[F((i+1)\Delta t) + F(i\Delta t)]$$

APPENDIX C

UNCERTAINTIES IN THE EXPERIMENTAL INVESTIGATION

The strict definition of the term "uncertainty" relates to an inaccuracy that is erratic and can only be treated on a statistical bases. In this sense at least one of the inaccuracies discussed in this appendix is not an uncertainty, since it is amenable to direct calculation. This error (item 2 below) is, however, an error in an intermediate physical quantity, and will result in an output error which will vary with the function being transformed. It is therefore treated as an uncertainty.

There are three general sources of uncertainty inherent in the experimental setup, and one source arising from the method chosen to evaluate the data. Since a thorough quantitative investigation of these sources was not attempted the listing is not necessarily in order of magnitude.

(1) Variation in the angle of rotation existing between adjacent strips.

(2) Uncertainties due to the fact that the function was not rotated about the actual axis of the spring.

(3) Vertical separations appearing between adjacent strips as projected into the component planes.

(4) Variations in photographic paper density.

Items (1) and (2) concern uncertainties effecting an intermediate quantity, the angle through which the strips have been rotated. Items (3) and (4) are inaccuracies in the integration process which produces the output quantity.

The variation in the angles between adjacent strips was measured for an extreme angle of twist. The average angle between strips was 14.2°, the

maximum 16.5° and the minimum 10°. The variation between adjacent groups of three angles, however, was much less, the minimum being 42° and the maximum 45°. This indicates that for a regular function such as those transformed most of the individual errors resulting from the non-uniform rotation would cancel out.

The second source of uncertainty listed occurred as a consequence of fastening the paper to one side of the spring. As a result the strips do not rotate about the axis of the spring, but instead are tangent to a small circle. The radius of this circle is equal to the distance between the axis of the spring and the axis of a strip. This distance was of the order of 0.02 in. for the particular combination of spring, adhesive tape, and paper strips used.

The gaps appearing between adjacent strips as seen in the projected planes are caused primarily by the distortion of the spring. It has been demonstrated (19) with a shaft of rectangular cross section that lines drawn on the face perpendicular to the axis are distorted when the shaft is twisted. Since the strips are fastened along the entire width of the spring, this distortion produces a shear force between the spring and the strips. The resultant action of the individual strip then becomes dependent on which part of the distorted face of the spring remains firmly bonded to the strip. The magnitude of this uncertainty can be estimated by the difference in deviation between the curves of Figs. 7 and 9. Since only the envelope of the projected area was used in determining the curves of Fig. 7 the effect of the distortion of the spring does not appear. Figure 9, however, was the result of eliminating the lighted area which appeared between adjacent strips. A direct determination of the uncertainty can not be made for two reasons. The transformed functions were not identical for the two figures, although they did possess the same general characteristics. There is also the possibility that part of the difference in deviations was due to the errors calculated in Appendix B.

Uncertainties arising from non-uniform density of photographic paper were partially investigated. A critical case, in which the differences between the positive and negative areas was small, was chosen for this purpose. The results obtained by the weighing method were compared with those obtained by carefully graphing the photographed areas

and integrating by counting squares. Results differed by less than two percent of the reference magnitude.

APPENDIX D

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